

Coherent (spin-)tensor fields on D=4 anti-de Sitter space

A. Yu. Segal* and A. A. Sharapov

Department of Physics, Tomsk State University, Tomsk 634050, Russia

Abstract

The coherent states associated to the discrete series representations $D(E_o, s)$ of $SO(3, 2)$ are constructed in terms of (spin-)tensor fields on $D = 4$ anti-de Sitter space. For $E_o > s + 5$ the linear space $\mathcal{H}_{E_o, s}$ spanned by these states is proved to carry the unitary irreducible representation $D(E_o, s)$. The $SO(3, 2)$ -covariant generalized Fourier transform in this space is exhibited. The quasiclassical properties of the coherent states are analyzed. In particular, these states are shown to be localized on the time-like geodesics of anti-de Sitter space.

1 Introduction

The $D = 4$ anti-de Sitter (AdS) space is the maximally symmetrical solution of the Einstein equations with the negative cosmological constant. This space arises as a consistent vacuum solution in various modern field theories coupled to gravity. Among them there are the extended $D = 4$ supergravities [3], supersymmetric theories of the Kaluza-Klein type [2], interacting theories of the higher-spin massless fields by Fradkin and Vasiliev [4],[5].

The symmetry Lie algebra of the AdS space, $so(3, 2)$, has unitary representations with bounded energy that is crucial for the existence of the particle interpretation, it is possible also to introduce the notions of massive and massless particles [6]-[10]. Besides, there exist two remarkable ‘singleton’ representations [11] having no direct analog in the flat space and which were conjectured by Fronsdal and Flato to play a role of ‘preons’ for the massive and massless particles [12].

All this indicates that, like the Minkowski space, the AdS space may play an important role for the particle physics and deserves a serious study.

The essential ingredient of a perturbative field theory is a construction of the complete basis of solutions for linearized field equations. In this paper we construct the explicit realizations for the Hilbert spaces of solutions for the free relativistic wave equations corresponding to the massive unitary irreducible representations (UIR) of the AdS group. The absence of the simple analog of the Fourier transform, as distinguished from Minkowski space case, makes this task not so evident. Our approach to the problem is based on the use of *coherent states method* [13], that allows, in particular, to present a general solution for the equations as well as provide it with the transparent quasiclassical interpretation.

*e-mail: segal@phys.tsu.tomsk.su

We will proceed as follows. In Sec. 2 we recall some basic facts about the geometry of AdS space and the positive energy representations of AdS group. Sec. 3 is devoted to the derivation of families of the coherent states (CS) associated with the massive representations of the AdS group embedded into the space of irreducible (spin-)tensor fields on AdS space. In Sec. 4 we introduce the reproducing kernel associated to the family of CS which is the central object for the proof of unitarity and irreducibility of the representations in the linear spaces spanned by the CS set. In Sec. 4 we discuss the quasiclassical properties of the CS family. In Conclusion we summarize the results and discuss some perspectives. Appendix contains basic formulas of $SO(3, 2)$ -spinor formalism.

2 Preliminaries

The four-dimensional AdS space can be realized as a one-sheet hyperboloid embedded into five-dimensional pseudoeuclidean space $\mathbf{R}^{3,2}$ with coordinates $\{y^A\}$, $A = 5, 0, 1, 2, 3$. The embedding is set by the equation

$$\eta_{AB}y^Ay^B = -R^2 \quad , \quad \text{diag} \quad \eta_{AB} = (- \ - \ + \ + \ +) \quad (1)$$

where $\mathcal{R} = -12R^{-2}$ is the scalar curvature of AdS space. The space has the topology of $\mathbf{R}^3 \times S^1$ and can be globally parametrized by intrinsic coordinates (t, \vec{y}) defined by the rule

$$y^0 + iy^5 = e^{it}Y \ , \quad Y = \sqrt{R^2 + \vec{y}^2} \ , \quad (2)$$

$$0 \leq t < 2\pi$$

The induced metric is

$$ds^2 = -Y^2 dt^2 + dy_i dy^i - Y^{-2} y_i dy^i y_j dy^j \quad (3)$$

The isometric embedding (1) allows one to give an extremely simple geometrical description for the variety of all geodesics of AdS space. Namely, each geodesic may be visualized as a line of intersection of some two-plane in $\mathbf{R}^{3,2}$, passing through the origin, with the surface of hyperboloid. Any two-plane, in turn, can be spanned by a pair of orthogonal nonzero five-vectors u and v . In what follows we will be interested only in the time-like geodesics. In this case one may assume that $u^2 = v^2 = 1$. It is convenient to combine the introduced vectors into the complex one $p = u + iv$ satisfying the relations

$$p^2 = 0 \ , \quad (p, \bar{p}) = -2 \quad (4)$$

Then the $SO(2)$ -rotation of the orthonormal frame (u, v) in the respective two-plane corresponds to the change of p by a phase factor: $p \rightarrow e^{i\varphi}p$, and thus the variety of all oriented two-planes is in one-to-one correspondence with the points of quotient space defined by the eqs.(4) and the equivalence relation

$$p \sim e^{i\phi}p \ , \quad \forall \phi \in \mathbf{R} \quad (5)$$

In order to fix an orientation of the two-planes one can impose the additional condition ¹

$$i(p_0 \bar{p}_5 - \bar{p}_0 p_5) > 0 \quad (6)$$

¹Actually, in view of (4) the absolute value of this expression is always greater then or equals 1.

The orientation of two-planes induces an orientation on the respective time-like geodesics that may be thought of as the choice of the time direction. The inverse orientation results from the complex conjugation: $p \rightarrow \bar{p}$.

The AdS group $SO(3, 2)^\dagger \sim Sp(4)/\{\pm 1\}$ is the connected component of the identity of the group of all pseudoorthogonal transformations of $\mathbf{R}^{3,2}$. It preserves the orientation of ‘time-like’ two-planes (4). The infinitesimal transformations of AdS group on hyperboloid (1) are generated by ten vector fields

$$\mathcal{L}_{AB} = y_A \partial_B - y_B \partial_A \quad (7)$$

The corresponding Hermitean generators L_{AB} of $so(3, 2)$ (in the above realization $L_{AB} = i\mathcal{L}_{AB}$) form the algebra

$$[L_{AB}, L_{CD}] = i(\eta_{AC}L_{DB} + \eta_{BD}L_{CA} - \eta_{BC}L_{DA} - \eta_{AD}L_{CB}) \quad (8)$$

It is easily seen that the AdS group acts transitively on the set of time-like geodesics: using a proper $SO(3, 2)^\dagger$ -transformation any vector p can be transferred to the form $\mathring{p} = (i, 1, 0, 0, 0)$. The stability subgroup of \mathring{p} coincides with $SO(3)$. Taking into account the equivalence rel.(5) one concludes that the set of time-like geodesics is isomorphic to the quotient space

$$\mathcal{F} = \frac{SO(3, 2)^\dagger}{SO(3) \times SO(2)} \sim \mathbf{R}^6 \quad (9)$$

and eqs.(4-6) present its covariant realization.

In the coordinates (2), the shift $t \rightarrow t + \epsilon$ is the symmetry transformation generated by the unique time-like Killing vector \mathcal{L}_{05} . Therefore, the coordinate t can be naturally identified with the AdS time, while \mathcal{L}_{05} may be regarded as the energy operator.

The positive-energy UIR of $SO(3, 2)^\dagger \sim Sp(4)/\{\pm 1\}$ denoted as $D(E_o, s)$ are classified by two quantum numbers: minimal energy E_o and spin s , $s = 0, 1/2, 1, \dots$. The unitarity requires [10]

$$E_o = s + \frac{1}{2}, \quad s = 0, \frac{1}{2}; \quad E_o \geq s + 1, \quad s \geq 1. \quad (10)$$

$D(1/2, 0)$ and $D(3/2, 1/2)$ correspond to Dirac singletons [11, 12]; the representations $D(2, 0)$ and $D(s + 1, s)$, for any s , describe massless particles; finally, the massive particles are associated with the choice $E_o > s + 1$.

All these representations are of the lowest weight type and, as a result, they have a quite simple structure. The lowest weight of $D(E_o, s)$ (ground state) is defined as the eigenstate for the energy and the third-projection-of-spin operators

$$L_{05}|E_o, s\rangle = E_o|E_o, s\rangle, \quad L_{12}|E_o, s\rangle = s|E_o, s\rangle \quad (11.a)$$

annihilated by the corresponding lowering operators

$$E_j^-|E_o, s\rangle = 0, \quad S^-|E_o, s\rangle = 0, \quad j = 1, 2, 3 \quad (11.b)$$

where

$$E_j^\pm = (L_{0j} \pm iL_{5j}), \quad S^\pm = (L_{13} \pm iL_{32})$$

$$[L_{05}, E_j^\pm] = \pm E_j^\pm, \quad [L_{12}, S^\pm] = \pm S^\pm$$

It is also required for the subspace of states with the minimal energy E_o , denoted as $|\Lambda\rangle$, to carry the irreducible spin- s representation of the rotation subgroup $SO(3) \subset SO(3, 2)^\dagger$.

In this paper we will be interested in the massive representations. In this case the full basis B of the representations $D(E_o, s)$ may be obtained by the successive action on $|\Lambda\rangle$ by the rising operators E_j^+ :

$$B = \{ |\Lambda\rangle, \quad E_i^+ |\Lambda\rangle, \quad E_i^+ E_j^+ |\Lambda\rangle, \quad \dots \} \quad (12)$$

These representations can be also characterized by the eigenvalues of the quadratic and quartic Casimir operators associated to the $so(3, 2)$ algebra [10]

$$\begin{aligned} C_2 &= \frac{1}{2} L_{AB} L^{AB} = E_o(E_o + 3) + s(s + 1) \\ C_4 &= W_A W^A = -s(s + 1)(E_o - 1)(E_o - 2) \end{aligned} \quad (13)$$

$$W_A = \frac{1}{8} \epsilon_{ABCDE} L^{BC} L^{DE}$$

With the given ground state (11) one may associate the family of coherent states forming the overcomplete basis in the space of $D(E_o, s)$ and defined by the rule

$$|g\rangle = g |E_o, s\rangle, \quad \forall g \in SO(3, 2)^\dagger \quad (14)$$

Actually, the different elements $g \in SO(3, 2)^\dagger$ do not all define the different states. The elements g which change the ground state only by a phase factor

$$g |E_o, s\rangle = \exp(i\alpha(g)) |E_o, s\rangle$$

form a subgroup commonly referred to as a *stability subgroup* of the state $|E_o, s\rangle$.

From the definition (11) it follows that when $s = 0$ the ground state is invariant under the action of the maximal compact subgroup $SO(3) \times SO(2)$ while for nonvanishing s it is reduced to $SO(2) \times SO(2)$ generated by L_{05} and L_{12} . Thus the set of CS is labelled by the points of the quotient space \mathcal{F} in the spinless case and $SO(3, 2)^\dagger / SO(2) \times SO(2) \sim \mathcal{F} \times S^2$, when $s \neq 0$.

In the next section we give an explicit realization for the CS in terms of irreducible (spin-)tensor fields on AdS space.

3 Coherent (spin-)tensor fields on AdS space

Let us first consider the case of spinless particle on AdS space transformed under representation $D(E_o, 0)$. In the coordinate representation (7) this particle is described by the complex scalar field $\Phi(y) \equiv \langle y^A | \Phi \rangle$, where $|\Phi\rangle \in D(E_o, 0)$. In accordance with (11), the ground state $\overset{\circ}{\Phi}(y) = \langle y^A | E_o, s \rangle$ arises as the solution for the following system of equations:

$$\begin{aligned} \langle y | L_{ij} | E_o, s \rangle &= \mathcal{L}_{ij} \overset{\circ}{\Phi}(y) = 0 \\ \langle y | L_{05} - E_o | E_o, s \rangle &= (\mathcal{L}_{05} - E_o) \overset{\circ}{\Phi}(y) = 0 \\ \langle y | E_i^- | E_o, s \rangle &= (\mathcal{L}_{i0} - i\mathcal{L}_{5i}) \overset{\circ}{\Phi}(y) = 0 \end{aligned} \quad (15)$$

Up to a multiplicative constant, these equations have the unique solution

$$\mathring{\Phi}(y) = (y, \mathring{p}) = (y^0 + iy^5)^{-E_o} = e^{-iE_o t} Y^{-E_o} \quad (16)$$

Acting on this state by all the $SO(3, 2)^\dagger$ transformations one gets the family of CS

$$\Phi(y) = (y, p)^{-E_o}, \quad p_A = G_A^0 + iG_A^5, \quad G_A^B \in SO(3, 2)^\dagger \quad (17)$$

where by definition the complex vector p_A obeys the conditions (4), (6).

It is pertinent to note that the constructed basis of states will remain overcomplete even though one restricts the possible values of p^A onto the ‘massive hyperboloid’ setting $p_A = (i, \frac{1}{m}p_\mu)$, where $\mu = 0, 1, 2, 3$ and p_μ is the real Lorentz vector. This vector may be regarded as the four-vector of momentum of the particle in AdS space, in particular, the state with minimal energy corresponds to the usual transition to the rest reference system $p_\mu = (m, \vec{0})$. In this Lorenz-covariant parametrization the CS for a spinless particle was originally considered by Fronsdal [7].

Let us now consider the spinor field on AdS space $\Psi_a(y) \equiv \langle y, a | \Psi \rangle$, $|\Psi\rangle \in D(E_o, 1/2)$, transformed under the action of $sp(4)$ by the law²

$$(L_{AB}\Psi)_a = i\mathcal{L}_{AB}\Psi_a + i(\Gamma_{AB})^b_a \Psi_b \quad (18)$$

Then the corresponding system of differential equations for the ground looks like

$$\begin{aligned} \langle y, a | E_j^- | E_o, 1/2 \rangle &= i(\mathcal{L}_{0j} - i\mathcal{L}_{5j}) \mathring{\Psi}_a + i((\Gamma_{0j} - i\Gamma_{5j}) \mathring{\Psi})_a = 0 \\ \langle y, a | L_{05} - E_o | E_o, 1/2 \rangle &= (i\mathcal{L}_{05} - E_o) \mathring{\Psi}_a + i(\Gamma_{05} \mathring{\Psi})_a = 0 \\ \langle y, a | S^- | E_o, 1/2 \rangle &= i(\mathcal{L}_{13} - i\mathcal{L}_{32}) \mathring{\Psi}_a + i((\Gamma_{13} - i\Gamma_{32}) \mathring{\Psi})_a = 0 \\ \langle y, a | L_{12} | E_o, 1/2 \rangle &= (i\mathcal{L}_{12} - \frac{1}{2}) \mathring{\Psi}_a + i(\Gamma_{12} \mathring{\Psi})_a = 0 \end{aligned} \quad (19)$$

There are two linearly independent solutions to this system. In order to present them in a transparent form, let us consider the constant $Sp(4)$ -spinors ζ annihilated by the spinning part of the energy lowering operators

$$i(\Gamma_{0j} - i\Gamma_{5j})\zeta = 0 \iff (\Gamma_0 - i\Gamma_5)\zeta = 0 \quad (20)$$

Since the rank of the $SO(3)$ -invariant matrix $(\Gamma_0 - i\Gamma_5)$ equals 2, the two basis solutions ζ^\pm of (20) may be chosen to satisfy: $\Gamma_{12}\zeta^\pm = \pm\zeta^\pm$. As the consequence of (20) one can also find

$$i\Gamma_{05}\zeta^\pm = -\frac{1}{2}\zeta^\pm \quad (21)$$

It is now evident that because $Sp(4)$ -transformations (18) act separately on coordinates and indices one of the solutions to the system (19) may be obtained as the tensor product of a proper scalar field (16) and the spinor ζ^+ , i.e.

$$\mathring{\Psi}(y) = (y, \mathring{p})^{-E_o - \frac{1}{2}} \zeta^+ \quad (22)$$

²All the details of the $SO(3,2)$ -spinor formalism are collected in Appendix.

The second solution is immediately derived from (22) if one observes that the idempotent matrix $\hat{y} = R^{-1}y_A\Gamma^A$ defines the invariant operator ³

$$[L_{AB}, \hat{y}] = 0, \quad \hat{y}^2 = 1 \quad (23)$$

such that

$$\overset{\circ}{\Psi}' = \hat{y} \overset{\circ}{\Psi} = (y, p)^{-E_o - \frac{1}{2}} \hat{y} \zeta^+ \quad (24)$$

automatically satisfies (19) if $\overset{\circ}{\Psi}$ does.

By applying to these states a general AdS-transformation one gets two appropriate families of CS for describing a spin-1/2 particle

$$\Psi(y) = (y, p)^{-E_o - \frac{1}{2}} \zeta, \quad \Psi'(y) = (y, p)^{-E_o - \frac{1}{2}} \hat{y} \zeta, \quad (25)$$

where

$$p_A \Gamma^A \zeta = 0, \quad \zeta_a = G^b{}_a \zeta_b^+, \quad G^a{}_b \in Sp(4) \quad (26)$$

In order to normalize these solutions we also put

$$\bar{\zeta} \zeta = \sqrt{2}i \quad (27)$$

The relationship between the spinor and vector representations of AdS group is established by the standard relation

$$G^a{}_c (\Gamma_A)^c{}_d G^d{}_b = (\Gamma_B)^a{}_b G^B{}_A \quad (28)$$

Let us finally turn to the spin-1 case. For the sake of explicit $SO(3, 2)$ -covariance any tensor field on AdS space will be treated as a restriction to the hyperboloid surface of the one defined on $\mathbf{R}^{3,2}$ and subject to the y -transversality condition. For instance, a vector field $\Phi_A(y)$ on $\mathbf{R}^{3,2}$ can be unambiguously restricted to the surface (1) if it obeys the condition

$$y^A \Phi_A(y) = 0 \quad (29)$$

The action of the infinitesimal generators L_{AB} of $SO(3, 2)$ on this field is given by

$$(L_{AB} \Phi)_C = i \mathcal{L}_{AB} \Phi_C + i \eta_{AC} \Phi_B - i \eta_{BC} \Phi_A \quad (30)$$

As is in the previous cases, the condition for the field $\Phi_A(y)$ to realize the ground state of $D(E_o, 1)$ leads to a set of equations, which now have the form

$$\begin{aligned} \langle y, A | E_j^- | E_o, 1 \rangle &= i(\mathcal{L}_{0j} - i \mathcal{L}_{5j}) \overset{\circ}{\Phi}_A + i \eta_{0A} \overset{\circ}{\Phi}_j - i \eta_{jA} \overset{\circ}{\Phi}_0 + \\ &+ \eta_{5A} \overset{\circ}{\Phi}_j - \eta_{jA} \overset{\circ}{\Phi}_5 = 0 \\ \langle y, A | L_{05} - E_o | E_o, 1 \rangle &= (i \mathcal{L}_{05} - E_o) \overset{\circ}{\Phi}_A + i \eta_{0A} \overset{\circ}{\Phi}_5 - i \eta_{5A} \overset{\circ}{\Phi}_0 = 0 \\ \langle y, A | S^- | E_o, 1 \rangle &= i(\mathcal{L}_{13} - i \mathcal{L}_{32}) \overset{\circ}{\Phi}_A + i \eta_{1A} \overset{\circ}{\Phi}_3 - i \eta_{3A} \overset{\circ}{\Phi}_1 + \\ &+ \eta_{3A} \overset{\circ}{\Phi}_2 - \eta_{2A} \overset{\circ}{\Phi}_3 = 0 \\ \langle y, A | L_{12} | E_o, 1 \rangle &= (i \mathcal{L}_{12} - 1) \overset{\circ}{\Phi}_A + i \eta_{1A} \overset{\circ}{\Phi}_2 - i \eta_{2A} \overset{\circ}{\Phi}_1 = 0 \end{aligned} \quad (31)$$

³In fact, this matrix may be used for the invariant definition of the right (left) handed Weyl spinors χ_+ (χ_-) on AdS space as spinors obeying the conditions $\hat{y} \chi_{\pm} = \pm \chi_{\pm}$. In this paper, however, only the Dirac spinors will be used.

After straightforward calculations one finds that the system (29),(31) possesses the unique (up to a constant) solution which may be written as follows:

$$\overset{\circ}{\Phi}_A(y) = (y, \overset{\circ}{p})^{-E_o-1} h_A \quad (32)$$

$$h_A = (y, \overset{\circ}{p}) \overset{\circ}{q}_A - (y, \overset{\circ}{q}) \overset{\circ}{p}_A \quad , \quad y^A h_A \equiv 0$$

where

$$\overset{\circ}{q} = (0, 0, i, 1, 0) \quad (33)$$

A coherent state of a general form is obtained by the replacement: $(\overset{\circ}{p}, \overset{\circ}{q}) \rightarrow (p, q)$, where by definition the latter pair of vectors is constrained to satisfy

$$q^2 = (q, p) = (q, \bar{p}) = 0 \quad , \quad (q, \bar{q}) = 2 \quad (34)$$

Rel. (28) makes it possible to reexpress the vector q via ζ and p as follows:

$$q^A = \frac{2}{3} \bar{p}_B \tilde{\zeta} \Gamma^{BA} \zeta \quad (35)$$

Now we are in a position to perform an explicit construction of CS for all massive representations of AdS group in appropriate spaces of higher rank (spin-)tensors. In order to do this, there is no need to write down and solve a respective set of equations for a ground state. The ground states for the higher-spin representations can be actually obtained by multiplying the ones for spin-1/2 and spin-1 particles

$$\overset{\circ}{\Phi}_{A_1 \dots A_n a_1 \dots a_m} = \overset{\circ}{\Phi}_{A_1} \dots \overset{\circ}{\Phi}_{A_n} \overset{\circ}{\Psi}_{a_1} \dots \overset{\circ}{\Psi}_{a_m} \quad (36)$$

(For the sake of simplicity we use here only the first-type solution for the spin-1/2 ground state. All subsequent results, however, can be readily reformulated in terms of Ψ' .) In so doing, the weight of the resulting ground-state is equal to the sum of weights of the multipliers. From the group-theoretical viewpoint this procedure is identified with the Young product of irreducible lowest-weight representations.

Thus, the most general family of CS obtained in such a way looks like

$$\Phi(p, \zeta | y)_{A_1 \dots A_n a_1 \dots a_m} = (y, p)^{-E_o-s} h_{A_1} \dots h_{A_n} \zeta_{a_1} \dots \zeta_{a_m} \quad (37)$$

where E_o is the minimal energy and

$$s = n + \frac{1}{2}m$$

is the spin. The fields (37) are symmetric in both groups of indices, traceless in its vector ones and possesses the Γ -transversality condition:

$$(\Gamma^{A_1})^{a_1}{}_a \Phi_{A_1 \dots A_n a_1 \dots a_m} = 0 \quad (38)$$

Also, the following equations hold true:

$$\left\{ i(\Gamma^{AB})^a{}_{a_1} \mathcal{L}_{AB} + E_o + \frac{1}{2}m \right\} \Phi_{A_1 \dots A_n a a_2 \dots a_m} = 0 \quad (39.a)$$

$$\left\{ \frac{1}{2} \mathcal{L}^{AB} \mathcal{L}_{AB} - (E_o + \frac{1}{2}m)(E_o + \frac{1}{2}m - 3) \right\} \Phi_{A_1 \dots A_n a_1 \dots a_m} = 0 \quad (39.b)$$

$$y^{A_1}\Phi_{A_1\dots A_n a_1\dots a_m} = 0 \quad , \quad \partial^{A_1}\Phi_{A_1\dots A_n a_1\dots a_m} = 0 \quad (39.c)$$

Eqs. (39) constitute the full set of relativistic wave equations for the irreducible massive fields on AdS space. Note that, when $m \neq 0$, the mass-shell condition (39.b) follows from (39.a). It should be stressed that for a given quantum numbers E_o, s only the first solution Ψ of (19) obeys these equations. This observation allows to remove the ambiguity in the choice of the ground state in the case $m \neq 0$. As a consequence, *the lowest weight state with given quantum numbers E_o, s is unique in the space of solutions for (39)*. For some special cases these equations present the AdS generalizations of the well-known wave equations for the flat space-time. For example, for $m = 0$, we have an ordinary equations for massive tensor fields; in the case $m = 1$ one may recognize the Rarita-Shwinger equations for half-integer spins; setting $n = 0$ we arrive at the Bargmann-Wigner equations, which for $m = 1$ is reduced to the equation for spin-1/2 particle in AdS space originally proposed by Dirac [1].

To conclude this Section, let us note that the relations (4-6), (26, 27), (34) imposed on the pair of parameters p, ζ (or p, q) define the ten-dimensional constrained surface being isomorphic to the group manifold $Sp(4)$ (or $SO(3, 2)^\dagger$). The phase transformations: $p \rightarrow e^{i\varphi}p$, $\zeta \rightarrow e^{i\psi}\zeta$, (or, respectively, $q \rightarrow e^{2i\psi}q$) preserve this surface and the corresponding physical states (they acquire only a phase factor). On the other hand, in Sec.2 it was argued that, for nonzero spin, the set of CS is in one-to-one correspondence with the points of eight-dimensional homogeneous space $\mathcal{F} \times S^2$. This implies that the equations (4-6), (26, 27), (34) supplemented by the equivalence relations

$$\zeta \sim e^{i\psi}\zeta \quad , \quad q \sim e^{2i\psi}q \quad , \quad \forall \psi \in \mathbf{R} \quad (40)$$

present two covariant realizations for this space. Hereafter, we will refer to $\mathcal{F} \times S^2$ as to the *dual space*.

4 Unitarity and reproducing kernel

In this section, we turn to the questions of irreducibility and unitarity of the AdS-group representations in the spaces $\mathcal{H}_{E_o, s}$ of fields spanned by the set of CS:

$$\varphi_{A_1\dots A_n a_1\dots a_m}(y) = \int_{\mathcal{F} \times S^2} \omega(p, \zeta) \tilde{\varphi}(p, \zeta) \Phi(p, \zeta|y)_{A_1\dots A_n a_1\dots a_m} \in \mathcal{H}_{E_o, s} \quad (41)$$

where the invariant eight-form looks like

$$\omega = (d\bar{\zeta}_a \wedge d\zeta^a) \wedge (d\bar{p}_A \wedge dp^A)^3 \quad (42)$$

and the coefficients $\tilde{\varphi}(p, \varsigma)$ satisfy the restrictions

$$\tilde{\varphi}(e^{i\alpha}p, e^{i\beta}\zeta) = e^{i\alpha(E_o + \frac{1}{2}m) - i\beta\frac{m}{2}} \tilde{\varphi}(p, \zeta) \quad (43)$$

for the integrand to be well-defined on the dual space $\mathcal{F} \times S^2$. In fact, the last relation tells us that the coefficients $\tilde{\varphi}$ belong to the space of special densities on $\mathcal{F} \times S^2$. It should be noted, however, that this space is too large to be isomorphic to $\mathcal{H}_{E_o, s}$. The corresponding isomorphic subspace is extracted by a projector to be specified further.

Hereafter we restrict our consideration to the case of integer energies E_o , then the quantum states of the particle will be described by single-valued wave functions for the

integer spins and double-valued for the halfinteger ones. Upon this restriction the massive representations are characterized by the inequality $E_o \geq s + 2$.

To assign $\mathcal{H}_{E_o, s}$ with a structure of a Hilbert space carrying a unitary representation of $SO(3, 2)^\dagger$ it is necessary to introduce an invariant positive-definite inner product. In this case, the irreducibility will follow directly from the uniqueness of the lowest weight state. This may be seen as follows. The unitarity causes $\mathcal{H}_{E_o, s}$ to be a direct sum of irreducible subspaces, on the other hand, the spectrum of the energy operator L_{05} is bounded from below in $\mathcal{H}_{E_o, s}$ by construction. Hence, each irreducible component possesses a unique lowest weight state and the number of components equals the number of ground states which is just one in the case at hand. Indeed, if there exist another lowest weight state $|E'_o, s'\rangle \in \mathcal{H}_{E_o, s}$ then, by construction, $E'_o \geq E_o$. As the values of two Casimir operators C_2 and C_4 (13) coincide for both weights (E_o, s) and (E'_o, s') one gets the equations

$$C_2(E_o, s) = C_2(E'_o, s') \quad , \quad C_4(E_o, s) = C_4(E'_o, s') \quad (44)$$

The cases when these equalities takes place are

$$(E'_o, s') = (E_o, s) \quad , \quad (s + 2, E_o - 2) \quad , \quad (45)$$

and those obtained from this two cases by the maps $E \rightarrow 3 - E$ and $s \rightarrow 1 - s$.

The first possibility corresponds only to the ground state as the consequence of the uniqueness of the lowest weight state with the given quantum numbers (see Sec. 2). In the case of massive representations, $E_o \geq s + 2$, the rest possibilities are ruled out because $E'_o \leq E_o$.

The relevant inner product reads

$$\langle \varphi_1 | \varphi_2 \rangle = N^{-1} \int_{AdS} \Omega \bar{\varphi}_1(y)_{A_1 \dots A_n a_1 \dots a_m} \varphi_2^{A_1 \dots A_n a_1 \dots a_m} \quad , \quad (46)$$

where

$$\Omega = -\frac{1}{24R} y^A \epsilon_{ABCDE} dy^B \wedge dy^C \wedge dy^D \wedge dy^E \quad (47)$$

is the invariant volume form on AdS space associated with the metric (3). The inner product is manifestly AdS-invariant but its positive-definiteness requires special study. With this in mind consider first the inner product of two CS. Substitution of (37) in (46) leads to the integral which converges only provided that $E_o + m/2 > 3/2$ and the result is

$$\begin{aligned} \langle p' \zeta' | p, \zeta \rangle &= \frac{\mu}{N} \frac{(\bar{\zeta}' \zeta)^m}{(\bar{p}', p)^{E_o + s}} [(\bar{p}', p)(\bar{q}', q) - (\bar{p}', q)(\bar{q}', p)]^n \\ \mu &= \frac{(-2)^{E_o + \frac{m}{2}} 2\pi^2 R}{E_o + s - 1} B\left(\frac{1}{2}, E_o + \frac{1}{2}m - \frac{3}{2}\right) \end{aligned} \quad (48)$$

It should be noted that due to the condition (6) this expression is well-defined for any two CS as (\bar{p}', p) can never come to zero. In order to normalize the states we put $N = (-1)^{E_o} 2^{n - E_o} \mu$. Then

$$0 < |\langle p' \zeta' | p, \zeta \rangle| \leq 1 \quad (49)$$

where the equality is reached only provided $p = p'$ and $\zeta = \zeta'$. We see that there are no two orthogonal states in the CS set.

One can come to the expression (48) from the following line of reasoning. First of all, it is sufficient to consider only the case when one of the wavefunctions is the ground state, i.e. to evaluate the function $F(\bar{p}, \bar{\zeta}) = \langle p, \zeta | \overset{\circ}{p}, \zeta^+ \rangle$, the general situation can then be restored by $SO(3, 2)^\dagger$ -transformations. Exploiting the invariance property of the inner product one may readily find that the function $F(\bar{p}, \bar{\zeta})$ realizes the lowest weight state in the space of complex densities on $\mathcal{F} \times S^2$ (43) or, what is the same, obeys the following system of equations:

$$\begin{aligned} \langle p, \zeta | E_j^- | \overset{\circ}{p}, \zeta^+ \rangle &= i(\mathcal{L}_{0j}^* - i\mathcal{L}_{5j}^*)F(\bar{p}, \bar{\zeta}) = 0 \\ \langle p, \zeta | L_{05} - E_o | \overset{\circ}{p}, \zeta^+ \rangle &= (i\mathcal{L}_{05}^* - E_o)F(\bar{p}, \bar{\zeta}) = 0 \\ \langle p, \zeta | S^- | \overset{\circ}{p}, \zeta^+ \rangle &= i(\mathcal{L}_{13}^* - i\mathcal{L}_{32}^*)F(\bar{p}, \bar{\zeta}) = 0 \\ \langle p, \zeta | L_{12} - s | \overset{\circ}{p}, \zeta^+ \rangle &= (i\mathcal{L}_{12}^* - s)F(\bar{p}, \bar{\zeta}) = 0 \end{aligned} \tag{50}$$

where

$$\mathcal{L}_{AB}^* = \bar{p}_A \bar{\partial}_B - \bar{p}_B \bar{\partial}_A + (\Gamma_{AB})^b{}_a \bar{\zeta}_b \bar{\partial}^a \tag{51}$$

Under the assumption of convergence of the integral (46), these equations enable one to determine F up to a multiplicative constant, which, in turn, may be fixed by the direct evaluation of the norm of the ground state $F(\overset{\circ}{p}, \bar{\zeta}^+)$.

The inner product of CS (48) is said to define the *reproducing kernel* of the CS family if it satisfies the equation

$$\int_{\mathcal{F} \times S^2} \omega(p, \zeta) \langle p', \zeta' | p, \zeta \rangle \langle p, \varsigma | p'', \zeta'' \rangle = c \langle p', \zeta' | p'', \zeta'' \rangle \tag{52}$$

for some finite constant c . The arguments similar to those that had been used to establish the explicit expression for the inner product (48) show that the equation (52) always holds true and the only troublesome moment is the convergence of the integral (52). It is equivalent to the condition

$$c = F(\overset{\circ}{p}, \bar{\zeta}^+)^{-1} \int_{\mathcal{F} \times S^2} \omega |F(\bar{p}, \bar{\zeta})|^2 < \infty \tag{53}$$

The function $c = c(E_o, s)$ is not easy to calculate explicitly. The rough estimation shows that the integral (53) converges when $E_o > s + 5$ and diverges if $E_o < s + 2$. In the former case, by making an appropriate renormalization of the kernel the constant c may be set to one and the relation (52) turns to the conventional condition for the kernel of the projector operator. This projector acts in the space of complex densities (43) and its image defines the AdS-invariant subspace, denoted $\mathcal{H}_{E_o, s}^*$, with the elements obeying the equation

$$\tilde{\varphi}(p' \zeta') = c \int_{\mathcal{F} \times S^2} \omega(p, \zeta) \langle p', \zeta' | p, \zeta \rangle \tilde{\varphi}(p, \zeta) \tag{54}$$

This is just the above-mentioned condition that we imposed on $\tilde{\varphi}$ in the definition of $\mathcal{H}_{E_o, s}$ (41). The inverse transformation formula

$$\tilde{\varphi}(p, \zeta) = \frac{1}{c} \int_{AdS} \Omega \bar{\Phi}(p, \varsigma | y)_{A_1 \dots A_n a_1 \dots a_m} \varphi(y)^{A_1 \dots A_n a_1 \dots a_m} \tag{55}$$

establishes the isomorphism between $\mathcal{H}_{E_o,s}$ and $\mathcal{H}_{E_o,s}^*$. Thus, the formulas (41), (54), (55) may be considered as a generalized Fourier transform for the massive fields over $D = 4$ AdS space.

Now it is straightforward to check that under the condition (53) the introduced above inner product actually defines a Hilbert space structure in $\mathcal{H}_{E_o,s}$. Indeed,

$$\begin{aligned} ||\varphi||^2 &= \int_{AdS} \Omega \bar{\varphi}(y)_{A_1 \dots A_n a_1 \dots a_m} \varphi^{A_1 \dots A_n a_1 \dots a_m} = \\ &= \int_{\mathcal{F} \times S^2} \omega(p', \zeta') \int_{\mathcal{F} \times S^2} \omega(p, \zeta) \bar{\varphi}(p', \zeta') \langle p', \zeta' | p, \zeta \rangle \tilde{\varphi}(p, \zeta) = c \int_{\mathcal{F} \times S^2} \omega | \tilde{\varphi} |^2 \end{aligned} \quad (56)$$

As a result, we have proved that for $E_o > s + 5$ the space $\mathcal{H}_{E_o,s}$ carries the unitary irreducible representation of $so(3, 2)$ algebra, and presented a generalized Fourier transform between $\mathcal{H}_{E_o,s}$ and $\mathcal{H}_{E_o,s}^*$, with explicit realization of corresponding Hilbert space structures.

It is worth to remark that the reproducing kernel considered as the set of functions $F_{p', \zeta'}(p, \zeta)$ generates an overcomplete basis of states in the space $\mathcal{H}_{E_o,s}^*$. In the special case $m = 0$ the CS of this type were previously considered in ref.[14] in the context of geometric quantization of a massive spinning particle on AdS space. In that paper the dual space $\mathcal{F} \times S^2$ was identified with the phase space of the massive spinning particle on AdS space. Besides this, not long ago the lagrangian model of AdS massive spinning particle was suggested, with the physical phase space being $\mathcal{F} \times S^2$ [15]. Therefore, the construction of this paper may be thought of as the coordinate representation realization for the quantum theory of the model of ref.[15].

5 Quasiclassical interpretation

In the previous section we have presented the coordinate-type realization for the quantum mechanics of the massive spinning particle in AdS space. The use of the CS technique provides one with explicit expression for the wave functions expanded via the overcomplete basis of states being invariant under the group action. Also the CS are known as the states possessing a minimal quantum uncertainty when one defines this notion in an invariant manner [13]. As the result, many of properties of these states turn out to be closely related to those for the classical theory. Very often this provides the most natural way to assign the classical system to their quantum analog.

Let us demonstrate that our CS are localized on the time-like geodesics of the AdS space. For this end, consider the norm of an arbitrary coherent state:

$$\begin{aligned} \langle p, \zeta | p, \zeta \rangle &= \int_{AdS} \Omega \rho(y) = 1 \\ \rho(y) &= \frac{(-1)^{\frac{m}{2}} 2^s}{N} \frac{(|y^A p_A|^2 - |y^A q_A|^2)^n}{|y^B p_B|^{2(E_o+s)}} \end{aligned} \quad (57)$$

Here ρ is proved to be non-negative function and plays the role of a probability density to find the particle at the space-time point y . Since any two time-like geodesics as well as CS are obtained from each other by a proper AdS-transformation, it is sufficient to establish the fact of localizability only for the states with minimal energy. The latter are associated with the following choice of the parameters:

$$p = \overset{\circ}{p} = (i, 1, 0, 0, 0), \Rightarrow q = (0, 0, \vec{q}) \quad (58)$$

Then the expression (57) takes the form

$$\rho_{E_o}(y) = \frac{(-1)^{\frac{m}{2}} 2^s (R^2 + (\vec{y}, \vec{n})^2)^n}{N (R^2 + \vec{y}^2)^{(E_o+s)}} \quad , \quad (59)$$

$$n^i = \frac{1}{2} \varepsilon_{ijk} q^j \bar{q}^k \quad , \quad \vec{n}^2 = 1 \quad .$$

Here we have passed to the local coordinates (\vec{y}, t) defined by (2). Now it is clear that the function ρ_{E_o} reaches the maximum at the points of the geodesic line $\vec{y} = 0$ which is naturally identified with the world line of the massive particle in the state of the rest. But this is exactly the geodesic associated to the vector $\overset{\circ}{p}$, as it was discussed in Sec.1. Thus, the complex vector p_A labels the family of CS and, simultaneously, the set of geodesics on which the corresponding coherent states are localized.

To assign a physical interpretation to the rest parameters let us introduce the following real five-vector:

$$N_A = \frac{1}{4} \varepsilon_{ABCDE} p^B \bar{p}^C q^D \bar{q}^E \quad , \quad (60)$$

$$p^A N_A = 0 \quad , \quad N^A N_A = 1.$$

The last relations imply that for a given point $p \in \mathcal{F}$, there are only two independent components of N spanning a unit two-sphere S^2 . The pair (p_A, N_B) represents the alternative covariant parametrization for the dual space $\mathcal{F} \times S^2$. Let us show that the introduced vector N has a direct physical interpretation as the vector collinear to the vector of spin. Really, in the rest frame (58) it takes the form $\overset{\circ}{N} = (0, 0, \vec{n})$, where \vec{n} is defined as in (59). Identifying the spin of the particle s with the absolute value of the total angular momentum $\vec{L} = \{\frac{1}{2} \varepsilon_{ijk} L^{jk}\}$ in the rest reference system one readily finds that

$$(\vec{n}, \vec{L}) \Phi(\overset{\circ}{p}, \bar{q}|y)_{A_1 \dots A_n a_1 \dots a_m} = s \Phi(\overset{\circ}{p}, \bar{q}|y)_{A_1 \dots A_n a_1 \dots a_m} \quad (61)$$

and thereby the unit vector \vec{n} is oriented along the spin of the particle.

As is seen from (59), the probability density decreases monotonously outward from the geodesic $\vec{y} = 0$. In the limit $|\vec{y}| \rightarrow \infty$ one finds the two types of asymptotical behavior: $\rho_{E_o} \sim |\vec{y}|^{2E_o+m}$ - along the spin direction, and $\rho_{E_o} \sim |\vec{y}|^{2(E_o+s)}$ - in the orthogonal directions. So, the extent of decreasing of ρ_{E_o} increases with increasing E_o . The last fact agrees well with usual physical notions: the greater the mass of the particle (E_o in our case) the smaller a region this particle is localized within.

6 Conclusion

The main results of this paper are the explicit realization for family of CS in the spaces of (spin-)tensor fields on AdS space and the construction, in the case $E_o > s+5$, of the Hilbert spaces $\mathcal{H}_{E_o, s}$ spanned by these states carrying the UIR $D(E_o, s)$ of AdS group. Also, a generalized Fourier transform is introduced which establishes the isomorphism between $\mathcal{H}_{E_o, s}$ and the space $\mathcal{H}_{E_o, s}^*$ of special densities on the dual space $\mathcal{F} \times S^2$ being the phase space of a massive spinning particle. For the rest interval of energies, $s+1 < E_o < s+5$, the reproducing kernel technique used in the paper does not provide a direct proof of the unitarity and irreducibility of AdS group representations in $H_{E_o, s}$. So, this case requires a special consideration.

The constructions of this paper could be relevant for the perturbative quantum field theory calculations on the AdS space background, in particular, the constructed family of CS may serve as a starting point for the covariant derivation of the propagator for the higher spin fields. For the low spins these propagators were derived in [16], but the method used there becomes rather cumbersome for $s > 2$.

We have considered quasiclassical properties of CS and observed the localizability of these states on the time-like geodesic lines of AdS space. This has allowed us to assign a physical interpretation for the CS parameters, namely, they have been shown to label the respective geodesic line and spin direction.

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8 Appendix. $SO(3, 2)$ -spinor formalism

The generating elements of Clifford algebra Γ_A are chosen to satisfy

$$\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = -2\eta_{AB} \mathbf{1} \quad , \quad \text{diag}(\eta_{AB}) = (- - + + +) \quad (62)$$

This algebra has the unique nontrivial irreducible representation of complex dimension 4. Besides, there exists a Majorana representation in which all Γ -matrices are purely imaginary. Matrices $(\Gamma_A)^a_b$, $a, b = 1, 2, 3, 4$, allow a simple realization in terms of ordinary four-dimensional Dirac γ -matrices, namely, one may put: $\Gamma_A = \gamma_A$, where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$.

16 matrices

$$\mathbf{1} \quad , \quad \Gamma_A \quad , \quad \Gamma_{AB} = -\frac{1}{4}[\Gamma_A, \Gamma_B] \quad (63)$$

constitute the full linear basis in $Mat(4, C)$.

The charge and Hermitian conjugation automorphisms defined through the matrices C and Γ are given by

$$\Gamma_A^t = C^{-1} \Gamma_A C \quad , \quad C^\dagger = C^{-1} \quad , \quad C^t = -C \quad ; \quad (64)$$

$$\Gamma^\dagger = -\Gamma \Gamma_A \Gamma^{-1} \quad , \quad \Gamma^\dagger = -\Gamma \quad , \quad \Gamma^2 = -\mathbf{1} \quad . \quad (65)$$

In representation, in which $\Gamma_0 \Gamma_5$ is Hermitean and Γ_i – anti-Hermitean, $i = 1, 2, 3$, one may put $\Gamma = \Gamma_0 \Gamma_5$. In the Majorana representation one has $C = \Gamma$.

The matrices C^{ab} and $(C^{-1})_{ab}$ are used for the rising and lowering of spinor indices.

There are following symmetry properties: $\Gamma_{AB} C$ are symmetric, C and $\Gamma_A C$ are anti-symmetric.

Matrices Γ_{AB} realize the spinor representation of $so(3, 2) \sim sp(4)$. Exponential map

$$G = \exp\left(\frac{1}{2} K^{AB} \Gamma_{AB}\right) \quad , \quad K_{AB} = -K_{BA} \quad , \quad \bar{K}_{AB} = K_{AB} \quad (66)$$

defines the spinor representation of $Sp(4)$, which is the double covering group of $SO(3, 2)^\dagger$, $SO(3, 2)^\dagger \sim Sp(4)/\pm \mathbf{1}$

Dirac and charge conjugated spinors are defined as

$$\bar{\psi} = \psi^\dagger \Gamma \quad , \quad \tilde{\psi} = \psi^t C^{-1} \quad (67)$$

The Majorana spinors are extracted by the condition $\bar{\psi} = \tilde{\psi}$.

The following useful formulas take place:

1. Fierz identity for two spinors

$$\bar{\vartheta} \otimes \psi = \frac{1}{4}(\bar{\vartheta}\psi)\mathbf{1} - \frac{1}{4}(\bar{\vartheta}\Gamma^A\psi)\Gamma_A - \frac{1}{2}(\bar{\vartheta}\Gamma^{AB}\psi)\Gamma_{AB} . \quad (68)$$

2. Contractions of Γ - matrices in vector indices

$$\begin{aligned} \Gamma_{ab}^A(\Gamma_A)_{cd} &= C_{ab}C_{cd} - 2(C_{ac}C_{bd} - C_{bc}C_{ad}) \\ \Gamma_{ab}^{AB}(\Gamma_{AB})_{cd} &= (C_{ac}C_{bd} + C_{bc}C_{ad}) \end{aligned} \quad (69)$$

$$\Gamma_{cm}^{AB}(\Gamma_B)_{nb} = \frac{1}{2} \left\{ (\Gamma^A)_{cn}C_{mb} + (\Gamma^A)_{cb}C_{nm} + (\Gamma^A)_{mn}C_{cb} + (\Gamma^A)_{mb}C_{nc} \right\}$$

3. Contractions of Γ - matrices in spinor indices

$$Tr(\Gamma_A \Gamma_B) = -4\eta_{AB} \quad (70)$$

$$Tr(\Gamma_{AB} \Gamma_{CD}) = \eta_{BC}\eta_{AD} - \eta_{AC}\eta_{BD} \quad (71)$$

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